PRACTICE MIDTERM 2 (VOJTA) - BRIEF SOLUTIONS

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(1) (a) (i) Let $y = (\sin(x))^x$
(ii) $\ln(y) = x\ln(\sin(x))$
(iii) By l'Hopital's rule:

$$\lim_{x\to 0^+} x\ln(\sin(x)) = \lim_{x\to 0^+} \frac{\ln(\sin(x))}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{\cos(x)}{\sin(x)}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x^2\frac{\cos(x)}{\sin(x)} = \lim_{x\to 0^+} -x(\cos(x))\frac{x}{\sin(x)}$$

However, $\lim_{x\to 0^+} \frac{x}{\sin(x)} = 1$ (by l'Hopital's rule again), $\lim_{x\to 0^+} \cos(x) = 1$, and $\lim_{x\to 0^+} -x = 0$, whence $\lim_{x\to 0^+} -x(\cos(x))\frac{x}{\sin(x)} = 0 \times 1 \times 1 = 0$

(iv) We just found that $\lim_{x\to 0^+} \ln(y) = 0$, so $\lim_{x\to 0^+} y = e^0 = 1$

(v) Hence $\boxed{\lim_{x\to 0^+}(\sin(x))^x = 1}$

(b) $\boxed{0}$ (this limit is of the form $0^\infty = 0$, which is not an indeterminate form at all!)

(c) By l'Hopital's rule:

$$\lim_{x\to\infty} \frac{\cosh^{-1} x}{\ln x} = \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x^2-1}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2}\sqrt{1-\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^2}}} = 1$$

I used the fact that $\sqrt{x^2} = |x| = x$ (since $x > 0$)

(2) (a) We know that: $\lim_{t\to 0} \frac{\sin(t)}{t} = 1$. In particular, letting $t = \sin(x)$, we get: $\lim_{x\to 0} \frac{\sin(\sin(x))}{\sin(x)} = 1$. And letting $t = \pi x$, we get: $\lim_{x\to 0} \frac{\sin(\pi x)}{\pi x} = 1$. Using those two facts and multiplying the numerator by $\frac{\sin(x)}{\sin(x)}$ and the denominator by $\frac{\pi x}{\pi x}$, we get:

$$\lim_{x\to 0} \frac{\sin(\sin(x))}{\sin(\pi x)} = \lim_{x\to 0} \frac{\frac{\sin(x)}{\sin(x)}\sin(\sin(x))}{\frac{\pi x}{\pi x}\sin(\pi x)} = \lim_{x\to 0} \frac{\sin(x)\frac{\sin(\sin(x))}{\sin(x)}}{\pi x\frac{\sin(\pi x)}{\pi x}} = \lim_{x\to 0} \frac{\sin(x)\times 1}{\pi x \times 1} = \frac{1}{\pi}\lim_{x\to 0}\frac{\sin(x)}{x} = \frac{1}{\pi}$$

(b) $\boxed{\frac{1}{\sqrt{1-x^2}}}$

(c) $\boxed{-\sin(e^{x^2})e^{x^2}(2x)}$

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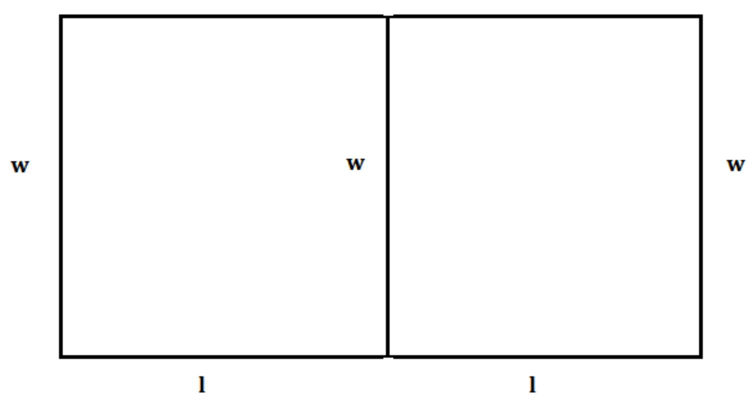
(3) (i) $y = x^{e^x}$
(ii) $\ln(y) = e^x \ln(x)$
(iii) $\frac{y'}{y} = \frac{e^x}{x} + e^x \ln(x)$
(iv) $\boxed{y' = x^{e^x}\left(\frac{e^x}{x} + e^x \ln(x)\right)}$

(4) $f'(x) = 0$ tells us that f is constant on each **piece** of its domain! Now because $0.5 \in (-\infty, 0)$ and $f(0.5) = 3$, we learn that $f(x) = 3$ on $(-\infty, 0)$. In particular, $\boxed{\lim_{x\to-\infty} f(x) = 3}$. Similarly, $3 \in (2, \infty)$, and $f(3) = 7$, so $f(x) = 7$ on $(2, \infty)$. In particular, $\boxed{\lim_{x\to\infty} f(x) = 7}$

(5) (i) See attached figure
(ii) We want to calculate the perimeter of the playground, which is $3w + 2l$. However, we also know that the total area, $2lw = 600$, so $w = \frac{300}{l}$, whence we get $f(l) = \frac{900}{l} + 2l$.
(iii) The only constraint is $l > 0$
(iv) $f'(w) = \frac{-900}{w^2} + 2 = 0 \Leftrightarrow w = \sqrt{450} = 15\sqrt{2}$
(v) It is easy to see that $f'(w) < 0$ for $w < 15\sqrt{2}$ and $f'(w) > 0$ for $w > 15\sqrt{2}$, so by the first derivative test for absolute extreme values (section 4.7), $w = 15\sqrt{2}$ is an absolute minimizer of f.
(vi) So the optimal dimensions of the playground are $\boxed{w = 15\sqrt{2}}$ and $\boxed{2l = \frac{600}{w} = 20\sqrt{2}}$ (see figure)

Note: If you thought that $lw = 600$, that's fine too, the question was kind of ambiguous. In this case, you should get $2l = 30\sqrt{2}$ and $w = 20\sqrt{2}$

1A/Practice Exams/Fence.png



(6) (D) Domain = All nonzero real numbers

(I) No x or y intercepts

(S) No symmetries

(A)
- **HA:** $\lim_{x\to\pm\infty} f(x) = 1$, so $y = 1$ is a H.A. at $\pm\infty$.
- Because of this, there are **no slant asymptotes**
- **VA:** $\lim_{x\to 0^+} f(x) = \infty$, $\lim_{x\to 0^-} f(x) = 0$, so there is a V.A. at $x = 0$.

(I) $f'(x) = e^{\frac{1}{x}}\left(\frac{-1}{x^2}\right) < 0$, so f is **decreasing** on $(-\infty, 0)$ and on $(0, \infty)$; No local maximums/minimums.

(C) $f''(x) = e^{\frac{1}{x}}\left(\frac{1}{x^4} + \frac{2}{x^3}\right) = e^{\frac{1}{x}}\left(\frac{1+2x}{x^4}\right)$, so f is concave down on $(-\infty, -\frac{1}{2})$ and concave down on $(-\frac{1}{2}, \infty)$; Inflection point = $(-\frac{1}{2}, e^{-2})$

The resulting graph looks somewhat like this:

1A/Practice Exams/Vojtagraph.png